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The Higgs model for anyons and Liouville action ⋆

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Abstract

We connect Liouville theory, anyons and Higgs model in a purely geometrical way.

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1. The Higgs model for anyons

The main aim of this paper is to show that well-known results [1,7] allow one to connect the Higgs model for anyons and Liouville theory within a rich geometrical structure. In a forthcoming paper [8] we will show how this geometrical framework allows us to obtain results concerning anyon spectra and the exclusion principle. The crucial point is based on the following remark:

Remark 1.

a. The configuration space of n anyons is the manifold of n unordered points in $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$,

$$M_n = (\widehat{\mathbb{C}}^n \backslash \Delta_n)/Symm(n), \tag{1.1}$$

with Δ_n the diagonal subset where two or more punctures coincide.

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b. The Liouville action on the Riemann sphere with n-punctures evaluated on the classical solution is the Kähler potential for the natural metric (the Weil–Petersson metric) on

$$\mathcal{M}_n = (\widehat{\mathbb{C}}^n \backslash \Delta_n)/Symm(n) \times PSL(2,\mathbb{C}). \tag{1.2}$$

This remark implies that starting from anyons on $\widehat{\mathbb{C}}$ one can recover the two-form associated to the natural metric on the configuration space by first computing the Poincaré metric $e^{\varphi_{cl}}$ on the punctured sphere and then, after evaluating the Liouville action for $\varphi = \varphi_{cl}$, computing the curvature two-form of the Hermitian line bundle on $\mathcal{M}_n$ defined by the classical action (see Section 5).

To understand the physical relevance of this remark we recall that the quantum Hamiltonian for n anyons is proportional to the covariant Laplacian on M_n.

Actually, the connection between anyons and Liouville arises also in considering the critically coupled Abelian Higgs model in (2+1)-dimensions where the space M_n plays a crucial role in the analysis of the nth topological sector of the theory. Remarkably $f = 2\mathrm{Re}\,\log\phi$, with ϕ the Higgs field, satisfies the modified (non covariant) Liouville equation

$$f_{z\bar{z}} = e^f - 1, \tag{1.3}$$

which is the second Bogomol'nyi equation. Notice that the non covariance (the -1 in (1.3)) is due to the Higgs mass.

In Refs. [1,2] Samols has studied the metric on vortex moduli space in the Abelian Higgs model. In Ref. [3] Manton has proposed a model for anyons which is based on the structure of such a space.

Let us consider the Lagrangian density of the Abelian Higgs model

$$\mathcal{L} = \tfrac{1}{2} D_\mu \phi \overline{D^\mu \phi} - \tfrac{1}{4} F_{\mu\nu} F^{\mu\nu} - \tfrac{1}{8}\left(|\phi|^2 - 1\right)^2, \qquad \phi = \phi_1 + i\phi_2, \tag{1.4}$$

where

$$D_\mu \phi = (\partial_\mu - iA_\mu)\phi, \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu,$$

and the metric has signature $(1,-1,-1)$.

In the temporal gauge the finiteness of the energy implies that at infinity the Higgs field is a pure phase. The magnetic flux through $\mathbb{R}^2$ is

$$\int F_{12} = 2\pi n, \tag{1.5}$$

where n is the winding number labelling the topological sectors of the map

$$|\phi| : S^1_\infty \longrightarrow U(1). \tag{1.6}$$

In the static configuration $\dot{A}_i = 0$, $\dot{\phi} = 0$, the energy has the lower bound $E \geq \pi|n|$. The critical case $E = \pi|n|$ arises when the Bogomol'nyi equations

$$(D_1 + \mathrm{sgn}(n) iD_2)\,\phi = 0, \qquad F_{12} + \mathrm{sgn}(n)\tfrac{1}{2}\left(|\phi|^2 - 1\right) = 0, \tag{1.7}$$

are satisfied. It is crucial that in the nth topological sector the space of smooth solutions is a manifold $\widetilde{M}_n$ of complex dimension n. In particular, each solution is uniquely specified by the n unordered points $\{z_k\}$ where the Higgs field is zero [4]. The same happens in considering the Liouville equation for the Poincaré metric on Riemann surfaces. In particular, due to the uniqueness of the solution of the Poincaré metric, to each complex structure of the n-punctured Riemann sphere (the unordered set of n points) corresponds a solution of the Liouville equation (see below for details).

Topologically $\widetilde{M}_n$ coincides with $\widehat{\mathbb{C}}^n$. It is a remarkable fact that the Abelian Higgs model (almost) provides a smooth metric and $U(1)$ gauge field on $\widetilde{M}_n$. In particular the kinetic energy induces the metric

$$ds^2 = \frac{1}{2}\int_{\mathbb{C}} (d\phi_a d\phi_a + dA_i dA_i). \tag{1.8}$$

The field evolution is described by geodesic motion on $\widetilde{M}_n$.

The z_k's are good coordinates only on the subspace M_n. Good global coordinates on M_n are provided by the coefficients of the polynomial [5]

$$P_n(z) = \sum_{k=0}^{n} w_k z^k \equiv \prod_{k=1}^{n}(z - z_k). \tag{1.9}$$

Note that the field evolution defines deformation of the complex structure of the punctured sphere. Recently it has been shown that this deformation is strictly related with Liouville theory (see below).

In Refs. [1,2] Samols introduced the following metric on $\widetilde{M}_n$:

$$ds^2 = \frac{1}{2}\sum_{r,s=1}^{n}\left(\delta_{rs} + 2\frac{\partial \bar{b}_r}{\partial z_s}\right) dz_r d\bar{z}_s, \tag{1.10}$$

where the b_r's satisfy the equations

$$\frac{\partial b_r}{\partial \bar{z}_s} = \frac{\partial \bar{b}_s}{\partial z_r}. \tag{1.11}$$

2. The Liouville equation

Let us denote by H the upper half-plane and by Γ a finitely generated Fuchsian group. A Riemann surface isomorphic to the quotient H/Γ has the Poincaré metric $\hat{g}$ as the unique metric with scalar curvature $R_{\hat{g}} = -1$ compatible with its complex structure. This implies the uniqueness of the solution of the Liouville equation on Σ. The Poincaré metric on H is

$$d\hat{s}^2 = \frac{|dw|^2}{(\operatorname{Im} w)^2}. \tag{2.1}$$

Note that $PSL(2,\mathbb{R})$ transformations are isometries of H endowed with the Poincaré metric.

An important property of Γ is that it is isomorphic to the fundamental group $\pi_1(\Sigma)$. Uniformizing groups admit the following structure. Suppose Γ uniformizes a surface of genus h with n punctures and m elliptic points with indices $2 \leq q_1^{-1} \leq q_2^{-1} \leq \ldots \leq q_m^{-1} < \infty$. In this case the Fuchsian group is generated by $2h$ hyperbolic elements $H_1, \ldots, H_{2h}$, m elliptic elements $E_1, \ldots, E_m$ and n parabolic elements $P_1, \ldots, P_n$, satisfying the relations

$$E_i^{q_i^{-1}} = I, \qquad \prod_{l=1}^{m} E_l \prod_{k=1}^{n} P_k \prod_{j=1}^{h} \left(H_{2j-1} H_{2j} H_{2j-1}^{-1} H_{2j}^{-1}\right) = I, \tag{2.2}$$

where the infinite cyclicity of parabolic fixed point stabilizers is understood.

Setting $w = J_H^{-1}(z)$ in (2.1), where $J_H^{-1} : \Sigma \to H$ is the inverse of the uniformization map, we get the Poincaré metric on Σ

$$d\hat{s}^2 = 2\hat{g}_{z\bar{z}}|dz|^2 = e^{\varphi_{cl}(z,\bar{z})}|dz|^2, \tag{2.3}$$

where

$$e^{\varphi_{cl}(z,\bar{z})} = \frac{|J_H^{-1}(z)'|^2}{(\operatorname{Im} J_H^{-1}(z))^2}, \tag{2.4}$$

which is invariant under $SL(2,\mathbb{R})$ fractional transformations of $J_H^{-1}(z)$. Since

$$R_{\hat{g}} = -\hat{g}^{z\bar{z}} \partial_z \partial_{\bar{z}} \log \hat{g}_{z\bar{z}}, \qquad \hat{g}^{z\bar{z}} = 2e^{-\varphi_{cl}}, \tag{2.5}$$

the condition $R_{\hat{g}} = -1$ is equivalent to the Liouville equation

$$\partial_z \partial_{\bar{z}} \varphi_{cl}(z,\bar{z}) = \tfrac{1}{2} e^{\varphi_{cl}(z,\bar{z})}. \tag{2.6}$$

Eq. (2.4) shows that from the explicit expression of the inverse map we can find the dependence of $e^{\varphi_{cl}}$ on the moduli of Σ. Conversely we can express the inverse map (to within an $SL(2,\mathbb{C})$ fractional transformation) in terms of φ_{cl}. This follows from the Schwarzian equation

$$\{J_H^{-1}, z\} = T^F(z), \tag{2.7}$$

where

$$T^F(z) = \varphi_{cl}'' - \tfrac{1}{2}(\varphi_{cl}')^2 \tag{2.8}$$

is the classical Liouville energy-momentum tensor and

$$\{f, z\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2 = -2(f')^{1/2}((f')^{-1/2})'' \tag{2.9}$$

is the Schwarzian derivative.

3. The Riemann sphere

Here we now discuss basic geometrical results on the Riemann sphere with n-punctures

$$\Sigma = \widehat{\mathbb{C}} \backslash \{z_1, \dots, z_n\}, \qquad \widehat{\mathbb{C}} \equiv \mathbb{C} \cup \{\infty\}. \tag{3.1}$$

Let $P_1, \dots, P_n$, $n \geq 4$, be the set of parabolic generators of Γ satisfying the constraint $P_1 \cdots P_n = 1$ and with the property that their parabolic fixed points $\{w_1, \dots, w_n\} \in \mathbb{R} \cup \{\infty\}$ map onto $\{z_1, \dots, z_n\}$. The moduli space of n-punctured spheres is the space of classes of isomorphic Σ's, that is

$$\mathcal{M}_n = \{(z_1, \dots, z_n) \in \widehat{\mathbb{C}}^n \mid z_j \neq z_k \text{ for } j \neq k\} / Symm(n) \times PSL(2, \mathbb{C}), \tag{3.2}$$

where $Symm(n)$ acts by permuting $\{z_1, \dots, z_n\}$ whereas $PSL(2, \mathbb{C})$ acts by linear fractional transformations. By $PSL(2, \mathbb{C})$ we can recover the 'standard normalization': $z_{n-2} = 0$, $z_{n-1} = 1$ and $z_n = \infty$. Furthermore, without loss of generality, we assume that $w_{n-2} = 0$, $w_{n-1} = 1$ and $w_n = \infty$. For the classical Liouville tensor we have

$$T^F(z) = \sum_{k=1}^{n-1} \left(\frac{1}{2(z - z_k)^2} + \frac{c_k}{z - z_k} \right). \tag{3.3}$$

Notice that T^F is holomorphic on $\Sigma = \widehat{\mathbb{C}} \backslash \{z_1, \dots, z_n\}$. Such a characteristic of T^F extends to higher genus surfaces as well. This follows from the Liouville equation. Equivalently one can consider local univalence of the inverse map, which implies that $\partial_z J_H^{-1}(z) \neq 0$, $\forall z \in \Sigma$, so that the Schwarzian derivative of J_H^{-1} is holomorphic on Σ.

The c_k's, called *accessory parameters*, satisfy the following constraints:

$$\sum_{k=1}^{n-1} c_k = 0, \qquad \sum_{k=1}^{n-1} c_k z_k = 1 - n/2, \tag{3.4}$$

so that $c_1, \dots, c_{n-3}$ can be considered as the basic set. These parameters are functions on the space

$$V^{(n)} = \{(z_1, \dots, z_{n-3}) \in \mathbb{C}^{n-3} \mid z_j \neq 0, 1; z_j \neq z_k, \text{ for } j \neq k\}, \tag{3.5}$$

which is a covering of $\mathcal{M}_n$ whereas the Teichmüller space T_n is a covering of $V^{(n)}$ whose transformations form a subgroup of the modular group. Note that

$$\mathcal{M}_n \cong V^{(n)} / Symm(n), \tag{3.6}$$

where the action of $Symm(n)$ on $V^{(n)}$ is defined by comparing (3.2) with (3.6). In particular, if a permutation involves at least one of the punctures between 0, 1 and ∞, then we must perform a linear fractional transformation to recover the standard normalization. This means that the last three punctures of the transformed surface $\widetilde{\Sigma}$

must be $\tilde{z}_{n-2} = 0$, $\tilde{z}_{n-1} = 1$, $\tilde{z}_n = \infty$. Thus if $\sigma_k \in Symm(n)$ interchanges z_k and z_{k+1}, the coordinate on $\tilde{\Sigma}$ is

$$\tilde{z} = \begin{cases} z, & k = 1, \ldots, n-4, \\ (z - z_k)/(1 - z_k), & k = n-3, \\ 1 - z, & k = n-2, \\ z/(z-1), & k = n-1. \end{cases} \tag{3.7}$$

4. Liouville action and the Weil–Petersson metric

Let us consider the Liouville action on the Riemann spheres with n-punctures [6]:

$$S^{(n)} = \lim_{r \to 0} S_r^{(n)} = \lim_{r \to 0} \left[\int_{\Sigma_r} (\partial_z \varphi \partial_{\bar{z}} \varphi + e^{\varphi}) + 2\pi (n \log r + 2(n-2) \log |\log r|) \right],$$

$$\Sigma_r = \Sigma \Bigg\backslash \left(\bigcup_{i=1}^{n-1} \{ z \mid |z - z_i| < r \} \cup \{ z \mid |z| > r^{-1} \} \right), \tag{4.1}$$

where the field φ is in the class of smooth functions on Σ with the boundary condition given by the asymptotic behaviour of the classical solution

$$\varphi_{cl}(z) = \begin{cases} -2\log|z - z_k| - 2\log|\log|z - z_k|| + \mathcal{O}(1), & z \to z_k,\ k \neq n, \\ -2\log|z| - 2\log\log|z| + \mathcal{O}(1), & z \to \infty. \end{cases} \tag{4.2}$$

Eq. (4.1) shows that already at the classical level the Liouville action needs a regularization, the effect of which is to cancel the contributions coming from the non covariance [1] of $|\varphi_z|^2$. This regularization provides a modular anomaly for the Liouville action, which is strictly related to the geometry of the moduli space [7]. In particular, it turns out that the Liouville action evaluated on the classical solution $S_{cl}^{(n)}$ is not invariant under the action of $Symm(n)$ [7],

$$S_{cl}^{(n)}(z_1, \ldots, z_{n-3}) - S_{cl}^{(n)}(\sigma_{i,n}(z_1, \ldots, z_{n-3}))$$
$$= \begin{cases} 4\pi \sum_{k \neq i} \log|z_k - z_i| - 2\pi(n-4)\log|z_i(z_i - 1)|, & i = 1, \ldots, n-3, \\ 4\pi \sum_{k=1}^{n-3} \log|z_k|, & i = n-2, \\ 4\pi \sum_{k=1}^{n-3} \log|z_k - 1|, & i = n-1, \end{cases} \tag{4.3}$$

where $\sigma_{i,n} \in Symm(n)$, $i \neq n$, is the transformation interchanging the i and n punctures. Furthermore, the asymptotic behaviour of the classical Liouville action when the punctures coalesce is [7]

[1] Recall that e^{φ} is a $(1,1)$-differential.

$$S_{cl}^{(n)}(z_1,\ldots,z_{n-3}) = \begin{cases} 2\pi\log|z_k-z_i|+\mathcal{O}(1), & z_i\to z_k,\ k\neq n,\\ 2\pi\log|z_i|+\mathcal{O}(1) & z_i\to\infty. \end{cases} \tag{4.4}$$

It turns out that the Liouville action computed on the classical solution is a continuously differentiable function on $V^{(n)}$ and [6]

$$-\frac{1}{2\pi}\frac{\partial S_{cl}^{(n)}}{\partial z_k} = c_k, \qquad k=1,\ldots,n-3, \tag{4.5}$$

where the c_k's are the accessory parameters defined in (3.3).

Notice that by (3.4) and (4.5) it follows that

$$\sum_{k=1}^{n-3}\frac{\partial S_{cl}^{(n)}}{\partial z_k} = 2\pi(c_{n-1}+c_{n-2}), \qquad \sum_{k=1}^{n-3} z_k\frac{\partial S_{cl}^{(n)}}{\partial z_k} = \pi(n-2+2c_{n-1}). \tag{4.6}$$

Since $S_{cl}^{(n)}$ is real, Eq. (4.5) yields

$$(\partial+\bar\partial)S_{cl}^{(n)} = -2\pi\sum_{k=1}^{n-3}(c_k dz_k + \bar c_k d\bar z_k), \tag{4.7}$$

and

$$\frac{\partial c_j}{\partial z_k} = \frac{\partial c_k}{\partial z_j}, \qquad j,k=1,\ldots,n-3, \tag{4.8}$$

$$\frac{\partial c_j}{\partial \bar z_k} = \frac{\partial \bar c_k}{\partial z_j}, \qquad j,k=1,\ldots,n-3. \tag{4.9}$$

We stress the strict similarity between (4.9) for the accessory parameters and the Samols equations (1.11).

Another important result in [6] is

$$\frac{\partial c_j}{\partial \bar z_k} = \frac{1}{2\pi}\left\langle\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_k}\right\rangle, \qquad j,k=1,\ldots,n-3, \tag{4.10}$$

where the brackets denote the Weil–Petersson metric on the Teichmüller space T_n projected onto $V^{(n)}$. Therefore by (4.5)

$$\frac{\partial^2 S_{cl}^{(n)}}{\partial z_j\partial \bar z_k} = -\left\langle\frac{\partial}{\partial z_j},\frac{\partial}{\partial z_k}\right\rangle, \qquad j,k=1,\ldots,n-3, \tag{4.11}$$

that is

$$\omega_{WP} = \tfrac{1}{2}i\bar\partial\partial S_{cl}^{(n)} = -i\pi\sum_{j,k=1}^{n-3}\frac{\partial c_k}{\partial \bar z_j}d\bar z_j\wedge dz_k, \tag{4.12}$$

where ω_{WP} is the Weil–Petersson two-form on $V^{(n)}$. Thus the Liouville action evaluated on the classical solution is the Kähler potential for the Weil–Petersson two-form defining the symplectic structure on $V^{(n)}$.

5. Geometric quantization

The symplectic structure considered above allows us to consider the following Poisson bracket relations [9]:

$$\{c_j, c_k\}_{WP} = 0, \qquad \{c_j, z_k\}_{WP} = \frac{i}{\pi}\delta_{jk}, \tag{5.1}$$

where the brackets are defined with respect to Weil–Petersson metric ω_{WP}. These relations suggest performing the geometric quantization of the space $\mathcal{M}_n$. To do this we must define a suitable line bundle. Let us consider the function [7]

$$f_{\sigma_{k,n}} = \frac{\prod_{j\neq k,\, j=1}^{n-3}(z_j - z_k)^2}{(z_k(z_k - 1))^{n-4}}, \qquad k = 1, \ldots, n-3,$$
$$f_{\sigma_{n-2,n}} = \prod_{j=1}^{n-3} z_j^2, \qquad f_{\sigma_{n-1,n}} = \prod_{j=1}^{n-3}(z_j - 1)^2. \tag{5.2}$$

Extension by the composition $f_{\sigma_1\sigma_2} = (f_{\sigma_1} \circ \sigma_2) f_{\sigma_2}$ defines a 1-cocycle $\{f_\sigma\}_{\sigma\in Symm(n)}$ of $Symm(n)$ [7]. Let us now consider the holomorphic line bundle

$$\mathcal{L}_n = V^{(n)} \times \mathbb{C}/Symm(n), \tag{5.3}$$

on $\mathcal{M}_n$ where the action of $\sigma \in Symm(n)$ is defined by $(x, z) \to (\sigma x, f_\sigma(x) z)$, $x \in V^{(n)}$, $z \in \mathbb{C}$. Since

$$\exp\left(\frac{S_{cl}^{(n)} \circ \sigma}{\pi}\right) |f_\sigma|^2 = \exp\left(\frac{S_{cl}^{(n)}}{\pi}\right), \tag{5.4}$$

it follows that $\exp(S_{cl}^{(n)}/\pi)$ is a Hermitian metric in the line bundle $\mathcal{L}_n \to \mathcal{M}_n$. By (4.5) $\exp(S_{cl}^{(n)}/\pi)$ has connection form $-2\sum_i c_i dz_i$ and, by (4.12), curvature two-form $-(2i/\pi)\omega_{WP}$ [7]. The covariant derivatives are

$$\mathcal{D}_k = \partial_{z_k} - \partial_{z_k}\frac{S_{cl}^{(n)}}{\pi} = \partial_{z_k} + 2c_k, \qquad \overline{\mathcal{D}}_k = \partial_{\bar{z}_k}. \tag{5.5}$$

In the geometric quantization the Hilbert states are sections of $\mathcal{L}_n$ annihilated by 'half' of the derivatives (polarization). The natural choice is

$$\mathcal{H} = \left\{\psi \in \mathcal{L}_n \mid \overline{\mathcal{D}}_k\psi = 0\right\}, \tag{5.6}$$

with inner product

$$\langle\psi_1|\psi_2\rangle = \frac{1}{n!(n-3)!}\int_{\mathcal{M}_n} d(WP) e^{-S_{cl}^{(n)}/\pi}\overline{\psi}_1\psi_2, \tag{5.7}$$

where

$$d(WP) \equiv \left(\bigwedge^{n-3} \tfrac{1}{2} i\bar{\partial}\partial S_{cl}^{(n)}\right), \tag{5.8}$$

which by (4.12) is the Weil–Petersson volume form. An alternative to (5.8) is to use the volume form $\bigwedge^{n-3}\omega$, where

$$\omega = \frac{1}{2}i\bar{\partial}\partial \log\det \left\| \frac{\partial^2 S_{cl}^{(n)}}{\partial \bar{z}_j \partial z_k} \right\| . \tag{5.9}$$

However, in general, the correspondence principle can be proved only if the scalar product is defined with respect to the volume form associated with the Kähler potential [10]. This means that in our case we must use the Weil–Petersson volume form.

We conclude the discussion on the geometric quantization of $\mathcal{M}_n$ by noticing that an interesting alternative for the polarization choice in the geometric quantization above is

$$\mathcal{H} = \left\{ \widetilde{\psi} \in \mathcal{L}_n \mid \mathcal{D}_k \widetilde{\psi} = 0 \right\} . \tag{5.10}$$

In this case the states and the classical Liouville action are related by

$$\partial_{z_k} S_{cl}^{(n)} = \pi \partial_{z_k} \log \widetilde{\psi} . \tag{5.11}$$

6. Anyons, Higgs and Liouville theory

A crucial aspect that should be further investigated concerns the classical–quantum interplay arising both in Liouville and anyon theories. In Refs. [11,12] it has been emphasized that the regularization arising at the classical level for the Liouville action is strictly related to the conformal properties both of quantum and classical (Poincaré metric) Liouville operators. A similar approach should be applied to anyons to provide a geometrical interpretation of the statistics. As in the case of conformal weights in 2D quantum gravity [12] we can consider anyons as elliptic points whose ramification index fixes the statistics.

The approach considered here makes it possible [8] to connect anyon theory with the geometrical approach to quantum gravity recently considered in Refs. [11,13,12].

Another interesting aspect is the connection between Liouville and Higgs. This provides a way to consider the Higgs model in a 2D gravity framework.

Finally we note that our results should be useful in investigating the underlying geometry [2] of the approach considered in Ref. [14].

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[2] A crucial object in Ref. [14] is the braid group $B_n = \pi_1(M_n)$.

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